PHYS 231 – Assignment #3

Due Monday, Oct. 30 @ 10:00 am

In class we showed that any complex number Z can be written in either of the following two forms:

$$Z = X + jY$$
$$Z = |Z|\, e^{j\phi}$$

To go between these two forms, you can use the following relationships:

$$|Z| = \sqrt{X^2 + Y^2}$$
$$\tan\phi = \frac{Y}{X}$$
$$X = |Z| \cos\phi$$
$$Y = |Z| \sin\phi$$

We also introduced one more quantity called the *complex conjugate* of Z, which is denoted Z^*. To find the complex conjugate of Z, you simply reverse the sign in front of all the j's. So, for example, if:

$$Z = X + jY = |Z|\, e^{j\phi}$$

then:

$$Z^* = X - jY = |Z|\, e^{-j\phi}.$$

The complex conjugate is useful because it can be used to calculate $|Z|$, the magnitude of Z. Consider, for example, the following product:

$$ZZ^* = (X + jY)(X - jY) = X^2 + Y^2 = |Z|^2 .$$

Alternatively, using the exponential form of Z:

$$ZZ^* = \left(|Z|\, e^{j\phi}\right)\left(|Z|\, e^{-j\phi}\right) = |Z|^2 .$$

In both cases we get that $ZZ^* = |Z|^2$.

1. Consider the complex number:

$$Z = \frac{1}{-2+3j} \tag{1}$$

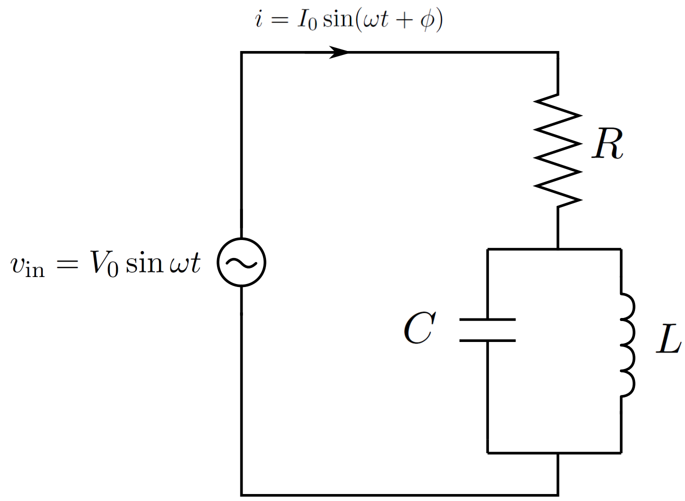
(a) Any complex number can be written in the form $Z = X + jY$ or $Z = |Z|e^{j\phi}$. Find X and Y for the complex number given in Eq. (1).

(b) Find the complex conjugate Z^*.

(c) Find $|Z|$ and $\tan\phi$ for this complex number.

(d) What is the value of ϕ in degrees?

2. Consider the LRC circuit below.



(a) Find an analytic expression for the equivalent impedance of the R, L, and C combination. Remember that $Z_C = 1/(j\omega C)$ and $Z_L = j\omega L$ and that you can combine impedances in series and parallel the same way that you would combine resistances. Give your answer in the form $Z = X + jY$ (i.e. find X and Y).

(b) The current in the resistor is given by $i = I_0 \sin(\omega t + \phi)$. Find I_0 and $\tan \phi$ in terms of V_0, ω, R, L, and C.

(c) Plot V_R/V_0 as a function of frequency. V_R represents the *amplitude* of the voltage across the resistor. Use component values $L = 300\ \mu\text{H}$, $C = 15\ \mu\text{F}$, and $R = 1000\ \Omega$ and plot over a frequency range of $f = 2200$–2500 Hz. Don't forget that $\omega = 2\pi f$. Don't draw your graph by hand, use plotting software.

(d) What is the impedance of the parallel LC part of the circuit when $\omega = 1/\sqrt{LC}$? Compare this to the impedance of a series combination of L and C when $\omega = 1/\sqrt{LC}$.

EXTRA Practice Problems – won't be graded.

- Eggleston Chapter 2: #9, 11

- Write $Z = \dfrac{3+6j}{(2-4j)^3}$ in the form $X + jY$ and $|Z|e^{j\phi}$.

Ans. $X = -\dfrac{21}{1000}$, $Y = -\dfrac{9}{125}$, $|Z| = \dfrac{3}{40}$, $\tan\phi = \dfrac{24}{7}$

- Find an expression for $\sqrt{j}$. Give you answer in the form $X + jY$ and $|Z|e^{j\phi}$. *Hint*: Use Euler's formula.

Ans. $\sqrt{j} = \pm\dfrac{1}{\sqrt{2}}(1+j) = \pm e^{j\frac{\pi}{4}}$

- Find an expression for $(-j)^{5/3}$. Give you answer in the form $X+jY$ and $|Z|e^{j\phi}$. *Hint*: Use Euler's formula.

Ans. $(-j)^{5/3} = -\dfrac{1}{2}\left(\sqrt{3}+j\right) = e^{-j5\pi/6}$

- Use Euler's equation to prove the trigonometric identity $\sin^2 x + \cos^2 x = 1$. *Hint*: Start with $e^{\pm jx} = \cos x \pm j\sin x$.

- Use Euler's equation to prove the following trigonometric identities:

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

Hint: Start with $e^{j(a\pm b)}$.

- Calculating the natural logarithm of negative numbers. First, show that $\ln Z = \ln|Z| + j\phi$ where Z is any complex number. Next, take $Z = -N$ (N is any positive real number) and, hence, show that:

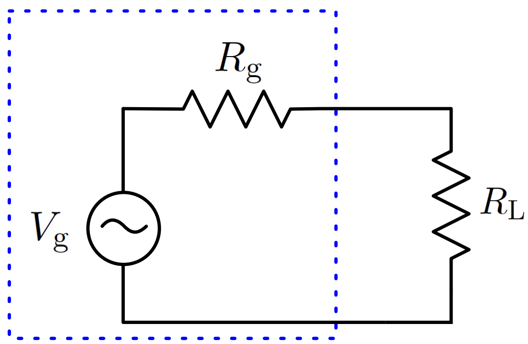
$$\ln(-N) = \ln N + j\pi(2k+1)$$

where k is an integer.

- Consider a resistor R in series with an inductor L driven by a function generator that outputs $v_{\text{in}} = V_0 \sin\omega t$. The current in the circuit is given by $i = I_0\sin(\omega t + \phi)$. Find I_0 and ϕ.

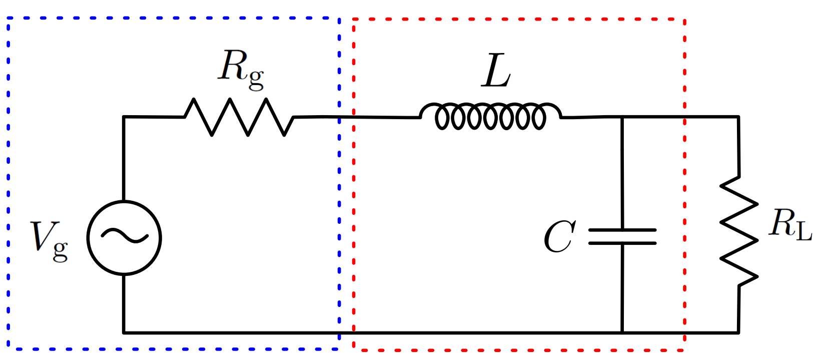
Ans. $I_0 = \dfrac{V_0/R}{\sqrt{1+\left(\omega\dfrac{L}{R}\right)^2}}$, $\tan\phi = -\omega\dfrac{L}{R}$

• Function generators, like the one you use in the lab, have a resistance R_{g} in series with the output of the generator. For example, the high-resistance output of the PASCO PI–9587C is 600 Ω. The figure below shows a schematic of a function generator (inside the blue dashed line) connected across a load resistance R_{L}.



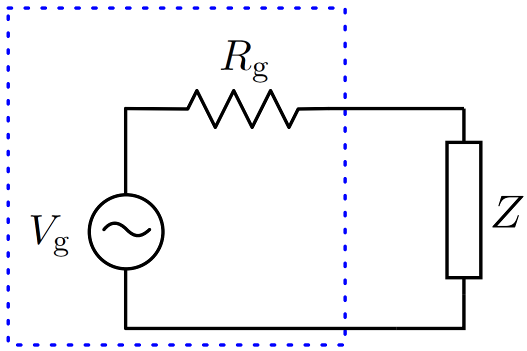
(a) For what value of R_{L} is the power dissipated by the load resistance a maximum? Express your answer in terms of R_g. That is, $R_{\mathrm{L}} = c\,R_{\mathrm{g}}$. Find the appropriate numerical value of the constant c. You must show your work/reasoning to earn full credit.

(b) If the values of R_{g} and R_{L} are not properly "matched", then we do not achieve maximum power transfer to the load which can be costly in high-power applications. One practical solution is to use a *matching network* to transform R_{L} such that the effective load as seen by the generator becomes equal to R_{g}. An example matching network, indicated by the red box, is shown below.



This circuit can be simplified by combining L, C, and R_{L} in a single impedance Z.
Show that Z can be expressed as:

$$Z = \frac{R_{\mathrm{L}}}{1 + (\omega R_{\mathrm{L}} C)^2} + j\omega L \left[\frac{1 + (\omega R_{\mathrm{L}} C)^2 - R_{\mathrm{L}}^2 \dfrac{C}{L}}{1 + (\omega R_{\mathrm{L}} C)^2} \right]$$



(c) Assume now that $R_\mathrm{L} = 2R_\mathrm{g}$. In order to achieve maximum power transfer to the effective load Z, we want to select C and L values such that $Z = R_\mathrm{g}$. For what values of C and L is $Z = R_\mathrm{g} = R_\mathrm{L}/2$? Hint: Set the imaginary part of Z equal to zero and the real part of Z equal to $R_\mathrm{L}/2$. Express your results for C and L in terms of R_L and ω.

This problem is an example of "impedance matching" using an LC matching network. We were able to transform the net load from a value of R_L to $R_\mathrm{L}/2 = R_\mathrm{g}$ so as to achieve maximum power transfer to the net load. You should have also noticed that the values of C and L required to achieve a match were frequency dependent. It is relatively easy to design matching networks to work well at one fixed frequency. Much more sophisticated circuits are required to design matching networks that function well over a wide range of frequencies. Some people make entire careers out of designing matching networks!